UNIVERSITY OF SASKATCHEWAN
EE815–Fundamentals of Wireless Communications
FINAL EXAMINATION, 2:00PM-5:00PM, April 12, 2007

Permitted Materials: Lecture Notes and Textbooks (Tse & Viswanath, Jafarkhani)

Examiner: Ha Nguyen

Note: **There are 5 questions. All questions are of equal value (with part marks indicated) but not necessarily of equal difficulty. Full marks shall only be given to solutions that are properly explained and justified.**

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1. (*Detection of on-off keying (OOK) in Rayleigh fading*) Consider the following input/output flat-fading channel model:

$$y = hx + w \tag{1}$$

where $w \sim \mathcal{CN}(0, N_0)$ represents AWGN, $h \sim \mathcal{CN}(0,1)$ represents the effect of Rayleigh fading and $x \in \{0, a\}$ is the binary information symbol to be transmitted. As usual, the two information symbols 0 and a are equally likely.

[5] (a) Consider *non-coherent* detection, i.e., the instantaneous value of h is unknown at the receiver. Develop the *maximum likelihood* (ML) detection rule and evaluate its exact performance in terms of SNR, where SNR is defined as

$$\mathsf{SNR} = \frac{\text{average received signal energy per complex symbol time}}{\text{noise energy per complex symbol time}} \tag{2}$$

[3] (b) Now consider *coherent* detection. What are the ML detection rule and its exact performance? Clearly explain your answers.
Hint: $E\left\{Q\left(\sqrt{2|h|^2\mathsf{SNR}}\right)\right\} = \frac{1}{2}\left(1 - \sqrt{\frac{\mathsf{SNR}}{1+\mathsf{SNR}}}\right)$.

[2] (c) Evaluate and compare the error probabilities of the above two detection schemes at $\mathsf{SNR} = 20$ dB.

2. (*Time-diversity codes*) Consider a Rayleigh flat-fading channel where the channel gains over different symbol durations (time slots) are i.i.d. $\mathcal{CN}(0,1)$. The transmit codewords $\mathbf{x} = [x_1, x_2]^\top$ are designed to have length 2. Furthermore, x_1 is drawn from a QPSK constellation while x_2 is determined by a "permuted" QPSK in the second time slot. The received signal is given by

$$y_l = h_l x_l + w_l, \quad l = 1, 2, \tag{3}$$

where w_1 and w_2 are $\mathcal{CN}(0, N_0)$. It is simple to see that, as far as the error performance is concerned, there are only two permutations of QPSK as shown in Figure 1.

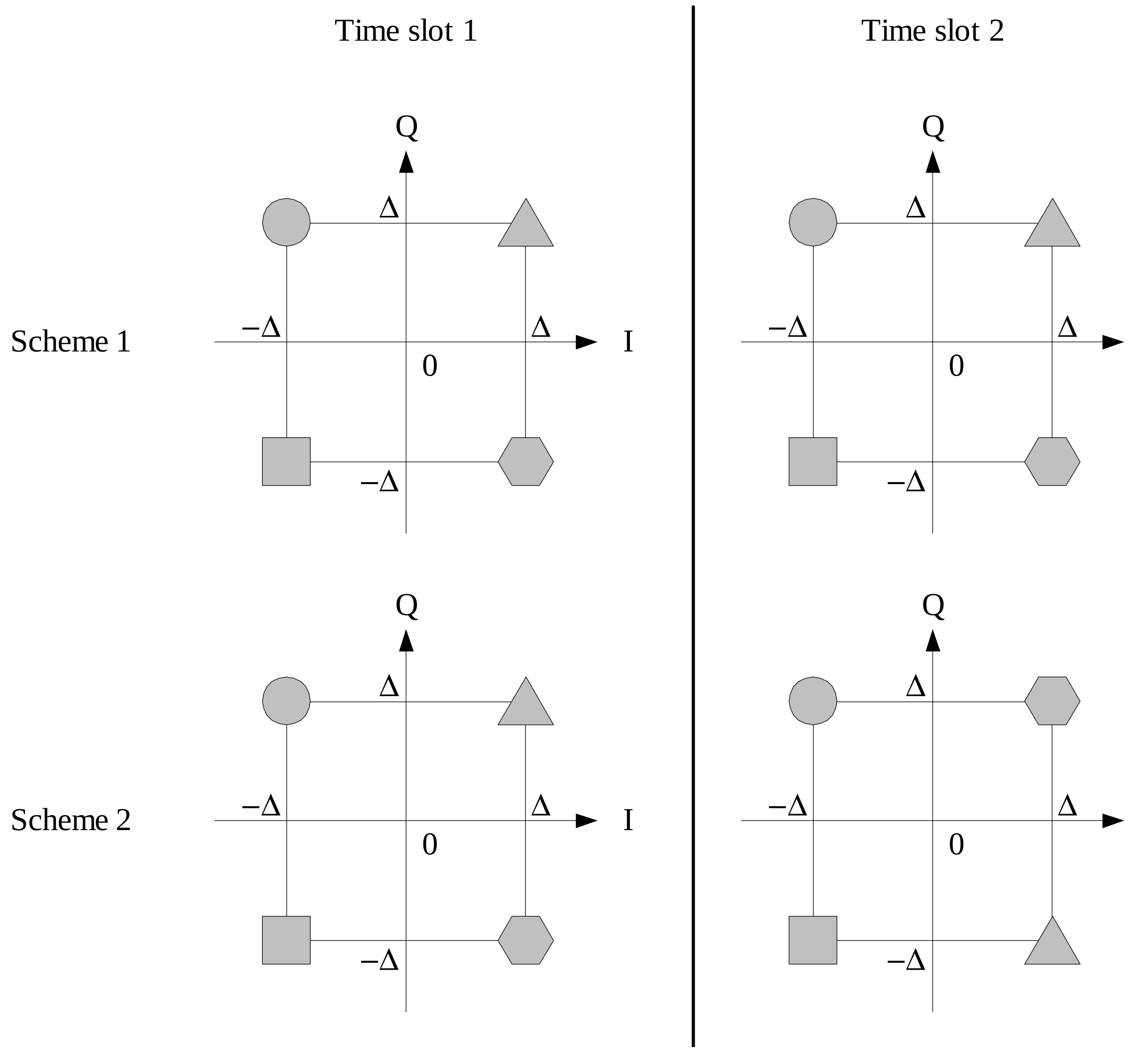

Figure 1: Two time-diversity codes based on QPSK.

[5] (a) Determine and compare the diversity gains and coding gains of the two schemes in Figure 1. Clearly explain your answers. *Hint:* You can assume that the codeword corresponding to the circle is transmitted. Also, you should not carry out the "worst case" analysis only.

[3] (b) Compare the diversity and coding gains of the better scheme found in (a) with the optimum rotation code for BPSK modulation presented in Section 3.2.2 of Tse & Viswanath's textbook. What is the comparison in terms of the code rate?

[2] (c) Suggest a method to improve over the permuted schemes considered in (a).

3. (*Space-time transmission*) Consider an i.i.d. Rayleigh flat-fading 2×1 MISO channel. You are presented with the following three matrices:

$$\mathbf{X}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{X}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{4}$$

[7] (a) Obtain the bounds on the pairwise error probabilities $P(\mathbf{X}_1 \to \mathbf{X}_2)$, $P(\mathbf{X}_1 \to \mathbf{X}_3)$ and $P(\mathbf{X}_2 \to \mathbf{X}_3)$ in terms of the SNR defined as in (2).

[3] (b) You are asked to make use of these matrices to transmit one information bit over two symbol periods. Which matrices you should use? Give your reasons.

4. (*Space-time code*) Consider the following 2×2 space-time code to be used over an i.i.d. Rayleigh flat-fading 2×1 MISO channel:

$$\mathbf{X} = \frac{1}{2}\begin{bmatrix} s_1 + s_2, & s_1 - s_2 \\ s_1 - s_2, & s_1 + s_2 \end{bmatrix}. \tag{5}$$

where s_1 and s_2 are information symbols drawn from a complex (two-dimensional) constellation Ω.

[3] (a) Show by giving an explicit example that the code is not orthogonal for an arbitrary constellation Ω.

[5] (b) Now consider Ω to be an M-PSK constellation. Show that that code is orthogonal. Also show that coherent ML detection of s_1 and s_2 can be decoupled by clearly giving the decision rules.

[2] (c) What is the diversity order of the code when Ω is an M-PSK? Explain your answer.

5. (*Universal code*) This question examines the universal property of Alamouti scheme. Consider the Alamouti transmit codeword $\mathbf{X} = \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix}$, where u_1 and u_2 are independent data symbols drawn from 2^R-QAM constellation.

[3] (a) For every codeword difference matrix

$$\mathbf{D} = \frac{\mathbf{X}_A - \mathbf{X}_B}{\sqrt{\mathsf{SNR}}} = \begin{bmatrix} d_1 & -d_2^* \\ d_2 & d_1^* \end{bmatrix} \tag{6}$$

show that the two singular values are the same and equal to $\sqrt{|d_1|^2 + |d_2|^2}$.

[3] (b) Assume that each of the QAM symbols u_1, u_2 is constrained in power of $\mathsf{SNR}/2$ (i.e., both the I and Q channels are peak constrained by $\pm\sqrt{\mathsf{SNR}/2}$), show that if the codeword difference d_l is not zero, then it is

$$|d_l|^2 \geq \frac{1/2}{2^R}, \quad l = 1, 2. \tag{7}$$

[2] (c) Determine the lower bound of the square of the smallest singular value of the codeword difference matrix and conclude that Alamouti scheme with uncoded QAMs on the two streams is approximately universal for the 2×1 channel.

[2] (d) Show that the Alamouti scheme is not universal for the 2×2 channel. Describe any universal scheme for the 2×2 channel that you know.

Potentially Useful Facts:

1. Given a square matrix $\mathbf{A}$. The *eigenvalues*, λ, of $\mathbf{A}$ are the roots of the following *characteristic equation* $\det[\mathbf{A} - \lambda \mathbf{I}] = 0$.
2. Suppose that $\mathbf{A}$ is an $N \times N$ matrix of rank $R_{\mathbf{A}}$. Then there exits a diagonal matrix $\mathbf{\Lambda} = \text{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$ and two unitary matrices $\mathbf{U}$ and $\mathbf{V}$ such that

$$\mathbf{A} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^H \tag{8}$$

   The above is known as the *singular value decomposition* of $\mathbf{A}$. The diagonal elements, σ_i, are called the *singular values* of $\mathbf{A}$. There are only $R_{\mathbf{A}}$ nonzero singular values. It can be shown that $\lambda_i = \sigma_i^2$ are the ith *eigenvalues* of $\mathbf{A}\mathbf{A}^H$.
3. For the real matrix $\mathbf{A} = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix}$, its SVD has the following form:

$$\begin{bmatrix} a & -a \\ -a & a \end{bmatrix} = \begin{bmatrix} -c & c \\ c & c \end{bmatrix} \cdot \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -c & c \\ c & c \end{bmatrix} \tag{9}$$